The struggle for beating Christofides: approximation of metric TSP

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Every student in Operations Research knows the Travelling Salesman Problem. Even its abbreviation TSP says enough for insiders. Solving the problem is NP-hard and it is a benchmark problem for testing new techniques invented for hard combinatorial optimization problems. Still, let us define the problem rigorously.

Introduction

Given a complete undirected graph $G = (V, E)$ on $n$ vertices with a length function, or cost function, on the edges $c \in \mathbb{R}_+^E$, find a shortest, minimum cost, tour that visits all the vertices of the graph exactly once and returns in the starting point. Such a tour is called a Hamilton Cycle in graph terminology. Thus, TSP is the problem of finding a shortest Hamilton Cycle in a complete graph.

The metric TSP is a natural restriction of TSP, in which the lengths on the edges satisfy the triangle inequality, i.e. when $c_{uv} + c_{vw} \geq c_{uw}$ for all $u, v, w \in V$. This problem students in Operations Research and Computer Science learn when studying approximation algorithms. In contradiction to the general TSP, the metric TSP admits a constant approximation ratio guarantee. It is easy to see that finding a Minimum Spanning Tree and doubling it gives an Euler graph of length no more than twice the length of an optimal TSP tour. Shortcutting it, due to the triangle inequality, gives a TSP-tour of no longer total length. In 1976, Christofides noticed that in order to get an Euler graph it is enough to match the vertices of odd degree in the minimum spanning tree only, adding only at most half the length of an optimal TSP tour. This gives an approximation ratio of $3/2$. Christofides thought that this was just a minor observation and never published it in a journal. Today, more than 35 years later, it is still the best approximation algorithm for metric TSP in terms of approximation guarantees. Improving it is seen as one of the prominent research challenges in combinatorial optimization.

Graph-TSP

In 2005, Gamarnik et al. were the first to make a small victory in the battle of improving on Christofides. The authors give a $1.487$-approximation for a very restricted case of the problem. Graph-TSP is a special case of metric TSP, where, given an undirected, unweighted underlying graph $G = (V, E)$, a complete weighted graph on $V$ is formed by dening the cost between two vertices as the number of edges on the shortest path between them. This new graph is known as the metric completion of $G$. Equivalently, this can be formulated as the problem of finding a spanning Eulerian multi-subgraph $H = (V, E')$ of $G$ with a minimum number of edges.

The result of Gamarnik et al. is on graph-TSP on 3-edge connected cubic graphs. A graph $G = (V, E)$ is cubic if all of its vertices have degree 3, and subcubic if they have degree at most 3. A graph is 3-edge connected if removal of any two edges keeps the graph connected. A bridge in a connected graph is an edge whose removal disconnects the graph. A graph is called simple if there is at most one edge between any pair of vertices. Gamarnik et al. gave a polynomial-time algorithm that finds a Hamilton cycle of cost at most $\tau n$ for $\tau = (3/2 - 5/389) \approx 1.487$ for graph-TSP on 3-edge connected cubic graphs. Since $n$ is the obvious lower bound for the optimal value for graph-TSP on such graphs, any tour of length $\tau n$, for any value of $\tau$, results in a $\tau$-approximation for the graph-TSP.
Figure 1 shows a cubic graph where the 3-2 ratio of Christofides’ algorithm is tight. Hence, this Christofides’ algorithm is not better than 3-2-approximate even when we restrict to cubic graphs.

**TSP Research**

The metric TSP is well-known to be NP-hard\(^1\). Even stronger, it is APX-hard, i.e., unless P=NP there exists some small number \(\gamma\) such that no polynomial time algorithm can approximate the TSP problem within a factor \(1 + \gamma\). This even holds for the graph-TSP problem \(^2\). It is unknown if this applies to cubic graphs as well. However, we do know that it is NP-hard since the Hamilton cycle problem is NP-complete in this case.

This was the state of the art when in 2010 Sylvia Boyd visited us at the Vrije Universiteit. Sylvia spent a significant part of her research on studying the TSP, and specifically a question on TSP tightly related to improving Christofides’ result. This related approach for finding approximated TSP solutions is to study the *integrality gap* \(\alpha(TSP)\), which is the worst-case ratio between the optimal solution for the TSP problem and the optimal solution to its linear programming relaxation, the so-called *Subtour Elimination Relaxation* (henceforth SER) (see 3 for more details), given by

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\begin{align*}
\min & \sum_{e \in E} c_e x_e \\
\text{s.t.} & \sum_{e \ni v} x_e = 2, & \forall v \in V; \\
& \sum_{e = \{u,v\}} x_e \geq 2, & \forall S \subset V; \\
& x_e \geq 0, & \forall e \in E.
\end{align*}
\]

The value \(\alpha(TSP)\) gives one measure of the quality of the lower bound provided by SER for the TSP. For metric TSP, it is known that \(\alpha(TSP)\) is at most 3/2 (see 4, 5), and is at least 4/3 (a ratio of 43 is reached asymptotically by the family of graph-TSP problems consisting of two vertices joined by three paths of length \(k\); see also 6 for a similar family of graphs giving this ratio), but the exact value of \((TSP)\) is not known. A constructive proof for value \(\alpha(TSP)\) would most likely provide an \(\alpha\) (TSP)-approximation algorithm for the TSP. There is the following well-known conjecture:

**Conjecture 1:** For the metric TSP, the integrality gap \((TSP)\) for SER is 4/3.

As with the quest to improve upon Christofides’ algorithm, the quest to prove or disprove this conjecture has been open for almost 30 years, with very little progress made.

Encouraged by the result of Gamarnik et al., during the visit of Sylvia we set ourselves the goal to settle the SER-conjecture for general cubic graphs, on the way deriving a 4/3-approximation algorithm for this subclass of graph-TSP. The paper that eventually emerged from our joint research studies the graph-TSP problem on cubic and subcubic graphs. Our main result indeed improves upon Christofides’ algorithm by providing a 4/3-approximation algorithm as well as proving 4/3 as an upper bound in Conjecture 1 for the the special case of graph-TSP for which the underlying graph \(G = (V,E)\) is a cubic graph.

Like Gamarnik et al.\(^7\), our approach is based on the following two observations:

1. Since \(n\) is a lower bound for the optimal value for graph-TSP as well as the associated SER\(^8\), it is enough to look for a polynomial-time algorithm that finds a Hamilton cycle of cost at most \(\tau n\) for some \(\tau < 3/2\). This gives a \(\tau\)-approximation for the graph-TSP, as well as a proof that the integrality gap \(\alpha(TSP)\) is at most \(\tau\) for these graphs.

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4. D. Shmoys and D. Williamson, 1990
5. L. Wolsey, 1980
8. To see that \(n\) is a lower bound for SER, sum all of the so-called “degree contraints” for SER. Dividing the result by 2 shows that the sum of the edge variables in any feasible SER solution equals \(n\).
2. A well-known theorem by Petersen\footnote{J. Petersen, 1891} states that every bridgeless cubic graph has a perfect matching.

Suppose that we remove a perfect matching from a given cubic graph. Then the remaining graph is a set of cycles which cover all vertices. If the number of cycles is \( k \) then we can form a TSP tour of length at most \( n + 2(k - 1) \) by connecting the cycles by a doubled spanning tree. The difficulty is to find a perfect matching that gives a cycle cover with a small number of cycles. We proved as a basic building block for our results the following theorem.

**Theorem 1:** Every bridgeless simple cubic graph \( G = (V, E) \) with \( n \geq 6 \) has a graph-TSP tour of length at most \( 4/3n - 2 \).

Our proof of this theorem is constructive, and provides a polynomial-time \( 4/3 \)-approximation algorithm for graph-TSP on bridgeless cubic graphs. The proof uses polyhedral techniques in a surprising way, which may be more widely applicable. The result also proves that Conjecture 1 is true for this class of TSP problems. The theorem is indeed central in the sense that the other results in our paper are based upon it. One of them is that we show how to incorporate bridges with the same guarantees.

For subcubic graphs it appeared to be harder to obtain the same strong results as for cubic graphs. For this class of graph-TSP we obtain a \( 7/5 \)-approximation algorithm and prove that the integrality gap is bounded by \( 7/5 \), still improving considerably over the existing \( 3/2 \) bounds. It is known that \( 4/3 \) is a lower bound for \( q(TSP) \) on subcubic graphs. We conjectured that \( 4/3 \) is the correct ratio to be obtained also for subcubic graphs.

Our paper was submitted to the IPCO 2011 conference in November 2010 and was accepted for presentation at the conference in June 2011 at the IBM Watson Labs in Yorktown Heights, USA (conferences like IPCO, SODA, STOC, FOCS typically have acceptance rates of 1 out of 4 papers). We have by now written a journal version.

**Other Works**

As often, rather intriguingly, happens, the problem seemed to have zoomed around in 2010. In January 2011, independent of our work, Aggarwal et al.\footnote{N. Aggarwal and N. Garg and S. Gupta, 2011} announced an alternative \( 4n/3 \) approximation for 3-edge connected cubic graphs only, but with a simpler algorithm. Their algorithm is based on the idea of finding a triangle- and square-free cycle cover, then shrinking and “splitting off” certain 5-cycles in the cover. Their result is restricted to 3-edge connected cubic graphs though.

Almost simultaneously with the previous paper, Gharan et al.\footnote{S.O. Gharan and A. Saberi and M. Singh, 2011} announced a randomized \( (3/2-\varepsilon) \)-approximation for graph-TSP for some tiny but strictly positive \( \varepsilon > 0 \). However, this is the very first polynomial-time algorithm with an approximation ratio strictly less than \( 3/2 \) for graph-TSP on general graphs. Their approach is very dierent from ours.

In the spring of 2011 a real breakthrough in this research area emerged. Mömke and Svensson\footnote{Tobias Mömke and Ola Svensson, 2011} came up with a powerful new approach, which enabled them to prove a \( 1.461 \)-approximation for graph-TSP for general graphs. In the context of the present paper it is interesting that their approach led to a bound of \( (4n/3 - 2/3) \) on the graph-TSP tour for all subcubic bridgeless graphs, thus improving upon our above mentioned \( (7n/5 - 4/5) \) bound and settling our conjecture armatively. Their approach was inspired by ours and also used our polyhedral technique.

In the meantime, end of 2011, the analysis of Mömke and Svensson was tightened by Mucha\footnote{Marcin Mucha, 2011} to attain a \( 13/9 \approx 1.44 \)-approximation ratio for general graph-TSP. In the very beginning of 2012, Sebő and Vygen\footnote{András Sebő and Jens Vygen, 2012} took a slightly different approach and managed to get the approximation ratio down to \( 7/5 \) for general graph-TSP.

The battle to beat Christofides has not been come to an end. At least after more than 30 years some progress has been made. There is a victory on graph-TSP. This is not a minor victory, since we suspect that graph-TSP provides the most difficult instances for metric TSP. But also on graph TSP, the battle is not over yet. The quest for a \( 4/3 \) approximation ratio and integrality gap for graph-TSP will continue.

For general metric TSP not even an \( \varepsilon \) has been scraped off the \( 3/2 \). Christodes’ \( 3/2 \) approximation still stands firmly and keeps challenging researchers in combinatorial optimization.

**References**

N. Aggarwal and N. Garg and S. Gupta: “A \( 4/3 \)-approximation for TSP on cubic 3-edge-connected graphs“, manuscript (2011)


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